$$M_n = (\frac{2n}{2n+1})^n \sqrt{1-\frac{2n}{2n+1}} = \left(\frac{2n}{2n+1}\right)^n \frac{1}{\sqrt{2n+1}}$$

d'où $\forall x \in [0,1]$, on a : $0 \le f_n(x) \le \left(\frac{2n}{2n+1}\right)^n \frac{1}{\sqrt{2n+1}}$.

Conclusion : f_n admet sur $[0,1]$ un maximum relatif égal à

$$M_n = \left(\frac{2n}{2n+1}\right)^n \frac{1}{\sqrt{2n+1}}.$$

b) On a : $0 \le x^n\sqrt{1-x} \le M_n \Rightarrow 0 \le U_n \le \int_0^1 M_n dx$

$0 \le U_n \le [M_n x]_0^1 \Rightarrow 0 \le U_n \le M_n$

$\lim\limits_{n\to+\infty} M_n = \lim\limits_{n\to+\infty} e^{n \operatorname{Log} \frac{2n}{2n+1}} \frac{1}{\sqrt{2n+1}} = 0$ car :

- $\lim\limits_{n\to+\infty} \frac{1}{\sqrt{2n+1}} = 0$
- $\lim\limits_{n\to+\infty} n\operatorname{Log}\frac{2n}{2n+1} = \lim\limits_{n\to+\infty} -n\operatorname{Log}\frac{2n+1}{2n} = \lim\limits_{n\to+\infty} -n\operatorname{Log}(1+\frac{1}{2n})$

$$= \lim_{X\to 0} \frac{-1}{2}\frac{\operatorname{Log}(1+X)}{X} = \frac{-1}{2}$$

d'ou (U_n) converge vers 0.

3°/a) $U_0 = \int_0^1 \sqrt{1-x}\, dx = \left[\frac{-2}{3}(1-x)^{\frac{3}{2}}\right]_0^1 = \frac{2}{3}$

b) $\begin{cases} u = x^n \\ v' = \sqrt{1-x} \end{cases}$ et $\begin{cases} u' = nx^{n-1} \\ v = -\frac{2}{3}(1-x)\sqrt{1-x} \end{cases}$

$$U_n = \left[-\frac{2}{3}(1-x)\sqrt{1-x}\cdot x^n\right]_0^1 + \frac{2}{3}n\int_0^1 x^{n-1}(1-x)\sqrt{1-x}\, dx$$

$$= \frac{2}{3}n\int_0^1 (x^{n-1}\sqrt{1-x} - x^n\sqrt{1-x})\, dx$$

$$U_n = \frac{2}{3}n\left(\int_0^1 x^{n-1}\sqrt{1-x}\, dx - \int_0^1 x^n\sqrt{1-x}\, dx\right).$$